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Statistical two-vortex equilibrium and vortex merger

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Two-vortex solutions from the mean field equations respecting all conservation laws of the Euler equations in two dimensions are calculated in a unit disk. These states are compared to symmetric and off-center single-vortex solutions. A thermodynamic stability analysis is used to determine their stability and a critical separation for stable two-vortex states is found from the crossover to instability. This could explain the merging critical ratios observed in numerical simulations and experiments.

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Vortex mergings are important in two-dimensional turbulence. For two-dimensional (2D) high-Reynolds-number fluids with random initial conditions, numerical simulations [1] have shown that beyond an early stage during which many coherent vortices are formed, the evolution is dominated by mergings of like sign vortices. A turbulence model has been proposed based on a description of vortex merging [2]. Many studies have been done on the merging of two identical vortices, as a first step to understand the evolution of turbulence. In numerical simulations different methods have been used to study the dynamics of two-vortex initial distributions [3,4]. A 2D pure electron plasma in a high magnetic field has also been used to model a 2D ideal fluid (the Euler equations) [5]. These dynamic studies find that two vortices will remain separate if their initial separation is greater than a critical value and below it they will quickly merge together. Equilibrium calculations for a pair of uniform vortices [6] are consistent with this picture because no steady state solutions are found for small vortex separations.

In this paper we look at the vortex merging problem using the mean field theory which respects all conserved quantities of the Euler equations [7]. The mean field statistical mechanics theory completely specifies the final equilibrium states by the values of the infinity of conserved quantities (integrals of any function of the vorticity), which in turn are determined by the initial condition. This approach has been used previously [8–10] to study single coherent vortices. Here we solve the two-vortex solutions of the equations in a disk and determine their thermodynamic stability properties. Some two-vortex solutions in a disk are found in Ref. [11] but they do not test their stability. The stability is crucial in explaining these states and mergers observed in experiments and simulations. A major assumption, as in any statistical mechanics, is that of ergodicity, i.e., that the dynamics samples all the phase space consistent with the conservation laws. There is increasing evidence [12] that for geometrically simple initial conditions the dynamics may not be mixing strongly enough for this assumption to be a good approximation. In the present work we separate the stability eigenvectors of a particular thermodynamic solution into those that might be expected to be strongly mixing, and those that are not. In particular the nonmixing eigenvectors take on rather small magnitudes, and in the infinite system limit correspond to collective symmetry motions of the vortex pair (translations and rotations). We then identify the passage of the eigenvalue of a "mixing" eigenvector through zero as indicating vortex merging in the dynamics.

The mean field equations for a two-dimensional ideal fluid is a set of differential-integral equations for a distribution function $n_0(\mathbf{r},\sigma)$ and the coarse-grained equilibrium vorticity field $\omega_0(\mathbf{r})$ is defined as $\int \sigma n_0(\mathbf{r},\sigma)d\sigma$. The equations depend on the infinite number of conserved quantities: $\int_{\eta(t)} f(\omega(\mathbf{r}))d\mathbf{r}$, for any path $\partial\eta(t)$ moving with the fluid and arbitrary function f. If we choose for simplicity a two-level (0 and q) initial vorticity field, the equations become [7]

$$\nabla^2\psi_0(\mathbf{r}) = -\omega_0(\mathbf{r}) = \frac{-q}{1+\exp[\beta(q\psi_0+\Omega r^2-\mu)]}. \quad (1)$$

Here $\psi_0(\mathbf{r})$ is the equilibrium stream function. Also μ, Ω, and β are constants to be determined by the total vorticity Q, angular momentum M, and energy E:

$$Q=\int \omega_0\, d\mathbf{r}, \quad M=\int r^2\omega_0 d\mathbf{r},$$

$$E=\frac{1}{2}\int \psi_0\omega_0 d\mathbf{r}.$$

The entropy is calculated as

$$S=-\int\left[\frac{\omega_0}{q}\ln\left(\frac{\omega_0}{q}\right)+\left(1-\frac{\omega_0}{q}\right)\ln\left(1-\frac{\omega_0}{q}\right)\right]d\mathbf{r}.$$

In the following calculations q is always set to one (a choice of normalization).

We use an iterative method to solve Eq. (1) in a unit disk with the boundary condition: $\psi_0(r=1)=0$. Starting with a guessed $\omega_t(\mathbf{r})$ with a two-vortex character, Poisson's equation $\nabla^2\psi_t=-\omega_t$ is solved. Substituting the stream function ψ_t into the right-hand side of Eq. (1), (μ,Ω,β) is found using a root-finding algorithm to get a new ω_t with required Q, M, and E. The calculation is then iterated until convergence. Similiarly we can also iterate at a fixed β and then calculate E afterward. Entropy could also replace energy as one of the prescribed quantities.

We plot two typical two-vortex distributions with $Q=0.2$ in Fig. 1. On the left two vortices are well separated and rotating around the disk center. The rotation frequency,

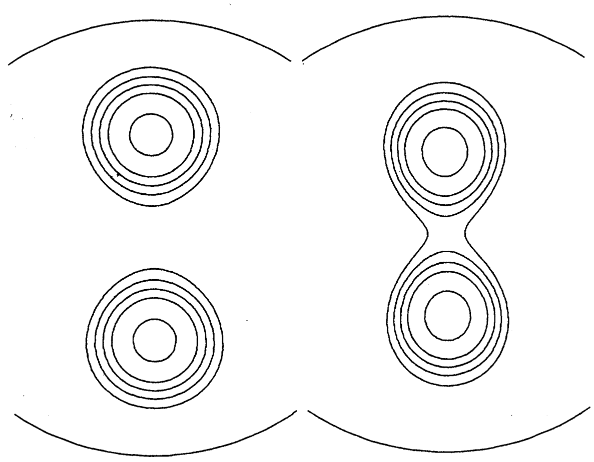

FIG. 1. Vorticity distributions for $Q=0.2$ and $S=0.25$. On the left graph, $M=0.05$; on the right, $M=0.0373$. Contour levels are 0.9,0.347,0.134,0.0518,0.02. Arcs indicate the disk boundaries.

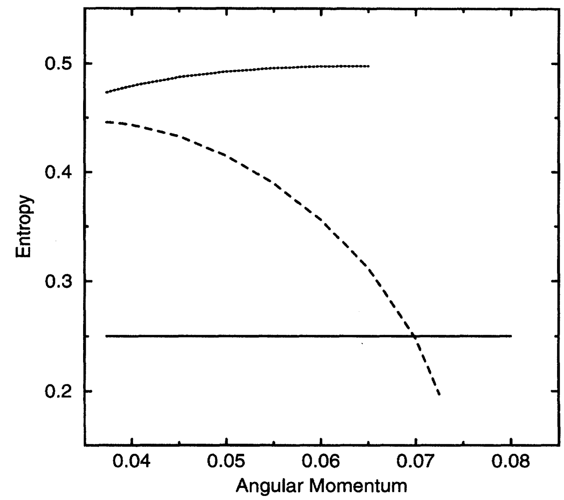

FIG. 2. Entropies for states with $Q=0.2$. Solid: two-vortex reference states. Dashed (symmetric vortex) and dotted (off-center vortex) lines have the same $E(M)$ as the solid line.

which is just 2Ω, is very close to the frequency of two point vortices sitting at the centers of mass of the vortices. As the two vortices get closer as shown on the right, the levels of contour lines joined together become larger. More vorticity is in the exchange band, so called because vorticity here will move back and forth in time between the two vortices. A numerical simulation [4] also observes this property. From Fig. 1 we also see that the aspect ratio (defined analogously to the aspect ratio of an ellipse) increases as the separation decreases, similar to previous calculations of equilibrium constant vorticity ellipses [6].

It is known that the mean field equations can have multiple solutions. In a disk two kinds of single vortices have been previous studied: symmetrical at the disk center and off-center [8,10]. In Fig. 2 we compare the thermodynamic properties of the two-vortex states with these single vortices. Usually solutions are described by E and S with fixed Q and M, but for two-vortex solutions it is more informative to fix Q and S and vary M to get similar states with different separations since M and the vortex separation are simply related. (E also changes with M.) These reference states are plotted in the figure as a solid line. We then use the physical parameters (Q,E,M) of each two-vortex solution on the solid line to find the corresponding symmetric and off-center vortices, plotted as the dashed and dotted lines respectively in Fig. 2. For two-vortex solutions, the vortex separation decreases with M and finally reaches a value below which no solutions could be found, i.e., the iterative algorithm failed to converge to a two-vortex solution and always converged to a single vortex. This suggests a critical separation for two vortices to merge, but could also reflect a failure of the root-seeking algorithm. Compared with symmetric vortices, the two-vortex states have higher entropies at large M, but the entropy becomes lower as M decreases. Lower entropy means a thermodynamically less probable state. But as long as they are local entropy maxima we could expect stable two-vortex states to rise in dynamical situations from nearby initial distributions. Off-center vortices (the dotted line) always have the largest entropies. We suspect that they are absolute entropy maxima for this system.

The mean field equations are obtained by requiring the vorticity distribution to be an entropy extremum. A question arises naturally whether a particular solution is a maximum, minimum, or saddle point. Only a maximum is expected to be stable. To determine the thermodynamic stability of a solution, a small variation $\delta n(\mathbf{r},\sigma)$ is added to a solution $n_0(\mathbf{r},\sigma)$ and the entropy change is examined. This idea has been used in Refs. [8,13] to calculate the stability of a single vortex in a disk for the point vortex model. It has also been used on the mean field theory to estimate the instability of a shear layer to a coherent vortex [9]. Here we use this idea to calculate the thermodynamic instability of two-vortex solutions. The change $\delta n(\mathbf{r},\sigma)$ of course needs to conserve Q, M, E, and all the other conserved quantities. In the case of a two-level initial vorticity field, the perturbation is just the addition of a small variation $\delta\omega(\mathbf{r})$ to $\omega_0(\mathbf{r})$ and $\nabla^2\delta\psi=-\delta\omega$. The changes in the total vorticity, angular momentum, and energy are

$$\delta Q=\int \delta\omega d\mathbf{r}, \quad \delta M=\int r^2\delta\omega\, d\mathbf{r}, \tag{2}$$

$$\delta E=\int (\psi_0\delta\omega+\tfrac{1}{2}\delta\psi\delta\omega)d\mathbf{r}\equiv\delta E^{(1)}+\delta E^{(2)}. \tag{3}$$

The entropy change, up to the second order, is

$$\delta S=\beta\int \psi_0\delta\omega d\mathbf{r}-\int \frac{(\delta\omega)^2}{2\omega_0(q-\omega_0)}d\mathbf{r}. \tag{4}$$

Since $\delta E=0$ we could use $\delta S-\beta\delta E$ instead of δS to get a quadratic form. Also because δS is only evaluated to the second order, for the energy constraint we only need to impose $\delta E^{(1)}=0$. To calculate these quantities we use the following procedure which could be used on any general 2D distribution. First $\delta\omega(\mathbf{r})$ is expanded in a complete set of

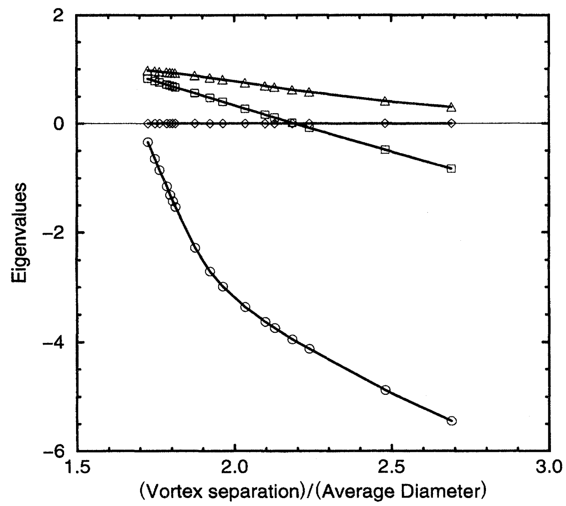

FIG. 3. Eigenvalues for different modes. Circle: even-real mode; diamond: even-imaginary; square: odd-real; triangle: odd-imaginary.

orthonormal functions $\phi_i(\mathbf{r})$: $\delta\omega(\mathbf{r}) = \Sigma_i a_i \phi_i(\mathbf{r})$. Then δQ, δM, $\delta E^{(1)}$, and $\delta S - \beta\delta E$ can be expressed as

$$\delta Q = \sum_i Q_i a_i, \quad \delta M = \sum_i M_i a_i,$$

$$\delta E^{(1)} = \sum_i E_i a_i, \quad \delta S - \beta\delta E = \sum_{ij} S_{ij} a_i a_j.$$

Here Q_i, M_i, and E_i are vectors and S_{ij} is a matrix depending only on $\omega_0(\mathbf{r})$. Now we make an arbitrary linear transformation transforming a_i to b_i but requiring $b_1 = \Sigma_i Q_i a_i$, $b_2 = \Sigma_i M_i a_i$, and $b_3 = \Sigma_i E_i a_i$. The constraints can now be satisfied with $b_1 = b_2 = b_3 = 0$. In this new coordinate system, S_{ij} changes to another matrix T_{ij} and $\delta S - \beta\delta E$ becomes $\Sigma_{ij>3} T_{ij} b_i b_j$. By removing the first three rows and columns from T_{ij}, we can perform the analysis without worrying about the constraints. If all the eigenvalues of the new T_{ij} are negative, the state (ω_0, ψ_0) will be an entropy maximum. When the largest eigenvalue reaches zero as system parameters change, the state becomes an entropy saddle point.

For the two-vortex solutions we expand $\delta\omega(\mathbf{r})$ in Fourier modes in the azimuthal direction and in Chebyshev polynomials in the radial direction:

$$\delta\omega(\mathbf{r}) = \sum_{m=-\infty}^{\infty} \sum_{\substack{n=0 \\ m+n=even}}^{\infty} a_{mn} \, \mathrm{T}_n(r) e^{im\theta}.$$

Here $m+n$ even is required to give the correct parity for each m mode. Because we have the two vortices sitting exactly at $\theta = 0$ and $\theta = \pi$, the expansion components of $\omega_0(\mathbf{r})$ are nonzero only for even m and all real. From Eqs. (2)–(4) we can see that for $\delta\omega$ this separates even m from odd m and also the real part of a_{mn} from the imaginary part. Thus the space for a_{mn} separates into four subspaces: even-real(cosine), even-imaginary(sine), odd-real(cosine), and odd-imaginary(sine) modes.

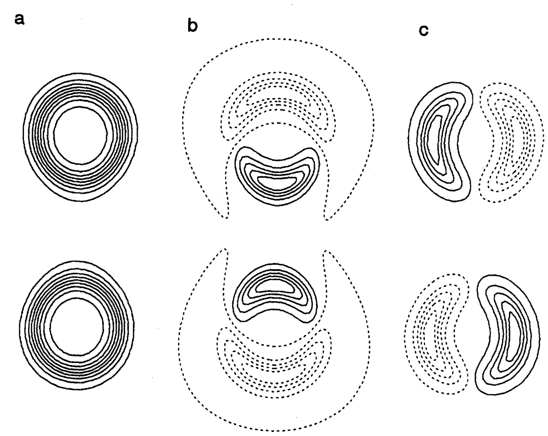

FIG. 4. (a) Vorticity distribution for $Q = 0.2$, $M = 0.0373$, and $S = 0.25$. (b) Eigenvector for the even-real mode. (c) Eigenvector for the even-imaginary mode. See text for contour levels.

In Fig. 3 we show the largest eigenvalue for each of the four subspaces for different values of M and fixed $Q = 0.2$ and $S = 0.25$ (the solid line in Fig. 2). The even-imaginary mode produces a rotation about the disk center [see Fig. 4(c)], and the eigenvalues (the diamonds) are zero as expected since the rotational symmetry persists in the finite system. The odd-real and odd-imaginary modes correspond to collective translational motions of the two vortices (along and perpendicular to the line joining the vortices) [14] which will give zero eigenvalues in the infinite system size limit. In the finite system the motion corresponding to these modes is not likely to be strongly mixing, so that the thermodynamic stability analysis may not be a reliable guide to the behavior in dynamics. In a pure electron plasma experiment [15] on two symmetric vortices in a disk, vortices are observed rotating around their symmetric equilibrium positions (with the equilibrium positions also rotating around the disk center). Their phases are such that the rotation is similar to a time-varying combination of the odd-real and odd-imaginary modes. Their result that the frequency and stability of the rotation could be explained by the dynamics of two point vortices confirms our conjecture about the nonmixing nature of these two modes. In a finite disk, the odd-imaginary mode moves the vortices closer by reducing their azimuthal separation with the corresponding eigenvalues (the triangles) always positive. This is not surprising since this motion is in the direction to form single off-center vortices which have larger entropies (see Fig. 2). The odd-real mode (the squares) has weakly positive eigenvalues at small M, and as M increases the eigenvalues decrease and then become negative. This can be understood because the vortices then sit closer to the boundary and feel its repulsive influence more.

The last mode is the even-real mode corresponding to a symmetric deformation of the two vortices. The eigenvalues of this mode take on larger magnitudes than the other modes, and we would expect the dynamics of this motion to be more strongly mixing. The eigenvalues (the circles in Fig. 3) are all negative. We identify the extrapolation of this eigenvalue to zero as indicating the merging of the vortex pair in the dynamics. This happens at a critical separation ratio (separation/average vortex diameter), which for the particular case in Fig. 3 occurs at about 1.7. This value lies in the range of values obtained by different papers [3–6].

We plot in Fig. 4 the mean field vorticity distribution and the even-real and even-imaginary mode eigenvectors of the leftmost point in Fig. 3. The state is plotted with nine contours from 0.1 to 0.9 with 0.1 increment. The ten contour levels for the eigenvectors are equally spaced with dotted lines for negative values. (The absolute amplitude is arbitrary.) We can clearly see that the effect of the even-real eigenvector in (b) is to move the two vortices closer, i.e., the mode produces a vortex merger. In fact, by adding the right combination of the even-real and even-imaginary eigenvectors to the mean field solution, we can get a vorticity distribution similar to the early stage of a two-vortex merger observed in a numerical simulation [4]. Finally we see (c) gives a small rigid-body rotation of (a).

It is reasonable to expect the critical ratio to be independent of the vortex size and this is normally the case in simulations and experiments. In the mean field theory, however, the vorticity field $\omega_0(\mathbf{r})$ is required to be a function of $\psi_c(\mathbf{r}) \equiv \psi_0(\mathbf{r}) + \Omega r^2$. For a two-vortex distribution ψ_c is peaked at the centers of the vortices and then, since $\Omega > 0$, after a minimum somewhere outside the vortices, increases for larger distances until cut off by the boundary. Thus for a large disk an unphysical vorticity ring is produced at the boundary and this ring will influence the stability calculation. For even larger disks the two-vortex solutions do not exist, but similiar states are still observed in experiments and simulations. We believe the reason is that in dynamics it is energetically unfavorable for vorticity to move across the minimum of ψ_c and the region outside will be unsampled. Hence in a free space two-vortex equilibrium could exist as a metastable state. We choose a small enough disk so that the boundary is not too far beyond the position of minimum ψ_c and the vortex ring is negligible. However the effect leads to a roughly 10% uncertainty on the critical ratio.

In conclusion we have calculated two-vortex mean field solutions in a disk. A thermodynamic stability analysis explains the existence of the stable two vortex states as local entropy maxima. Critical separations for their existence are also found from the crossing of the eigenvalues to positive values and with results consistent with previous work.

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